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Initial and boundary value problems of internal gravity waves

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The paper considers the generation of Boussinesq internal waves in the framework of the Green's function method. For certain domains it is shown how to construct Green's functions using the fundamental solution of the equation. The behaviour of the solution at large times for an impulsively started monochromatic point source is studied, attention being focused on the growth rate of the oscillation amplitude on the characteristic surfaces of the steady-oscillation equation which are emitted from the point source. In addition a simple extended source is considered, for which a focusing singularity phenomenon is shown to take place.

1. Introduction

Problems concerning motions of stratified fluids have attracted the attention of researchers for a long time. Many publications are devoted to these problems, and it is impossible to give an exhaustive review of these works in one paper. Detailed bibliographies on this subject can be found in the books by Lighthill (1978), Miropol'skii (1981), and Gill (1982). When investigating linear harmonic waves, researchers usually restrict themselves to considering equations for steady oscillations, probably because such an approach still enables one to obtain many interesting results. Among these results, the most interesting are those which are related to the phenomenon described in the two-dimensional case as a 'St. Andrew's Cross', consisting in the appearance of singularities in the amplitude of steady oscillations on certain lines or surfaces. Pictures obtained in the experiments of Mowbray & Rarity (1967) have become a classical illustration of this phenomenon. Mathematically the presence of singularities is explained by the well-known fact that solutions of hyperbolic equations can have singularities on their characteristic surfaces (see Hadamard 1932). Therefore if the steady oscillation equation is hyperbolic, we can expect the phenomenon mentioned. The peak of investigations concerning the presence of singularities in the steady oscillation amplitude was in the second half of the sixties and the beginning of the seventies. We mention here just the works of Wood (1965), Baines (1967), Larsen (1969) and Devanathan & Ramachandra Rao (1973). Among the further investigations based on the equations of steady oscillations and concerning the presence of singularities on characteristics, we point out those which are devoted to the case of variation with altitude of the buoyancy frequency (see Gordon, Klement & Stevenson 1975 and Liu, Nicolau & Stevenson 1990).

In the works mentioned, however, far less attention was paid to how the singularities arise in the amplitude. In this connection the work of Hendershott (1969) should be mentioned, where oscillations of a rotating stratified fluid caused by a pulsating sphere were investigated. It was shown in this work that on the characteristic cones tangent to the sphere the vertical component of the velocity field diverges as $t^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\sigma t}$ when $t \to \infty$, where σ is the frequency of pulsations. More precise investigations by Appleby & Crighton (1987) showed that at the apexes of these cones the divergence is as $t\,\mathrm{e}^{\mathrm{i}\sigma t}$. They named this phenomenon a 'focusing singularity'.

The present work aims to consider for the non-steady case the problem of the appearance of singularities in the amplitude. We proceed from the non-steady scalar equation for the vertical component of the velocity field:

$$(\partial^2/\partial t^2)\,\nabla^2 u + N^2(x_3)\,\hat{\nabla}^2 u = f(x,t), \tag{1.1}$$

$$\nabla^2 \equiv \partial^2/\partial x_3^2 + \hat{\nabla}^2, \quad \hat{\nabla}^2 \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2, \quad x = (x_1, x_2, x_3),$$

where $N(x_3)$ is the buoyancy frequency and $f(x,t)$ is a forcing. Equation (1) is known as the equation of internal gravity waves in the Boussinesq approximation. Its derivation can be found, for example, in the books by Gill (1982), Brekhovskikh & Goncharov (1982), and Whitham (1974).

The paper is organized as follows. In the next section results on representing solutions of initial and boundary value problems for equation (1.1) through Green's functions of relevant problems are presented. Formulae expressing Green's functions through the fundamental solution of (1.1) for some domains are given here as well. In §3 these results are applied to non-steady problems with monochromatic forcing. It is shown that, for an impulsively started monochromatic point source whose frequency σ is less than the buoyancy frequency, on the characteristic surfaces emitted from the source the vertical component of the velocity field diverges as $t^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\sigma t}$ for $t \to \infty$.

In addition, a problem with an extended source is considered, for which a phenomenon of the focusing singularity kind is shown to take place.

2. Initial and boundary value problems

Let $\epsilon^h(x,t)$ denote the function depending on variables (x,t) and parameter h, and satisfying the equation

$$\frac{\partial^2}{\partial t^2}\nabla^2 \epsilon^h(x,t) + N^2(x_3)\,\hat{\nabla}^2 \epsilon^h(x,t) = \delta(\hat{x}, x_3 - h, t) \tag{2.1}$$

where $x = (x_1, x_2, x_3)$ are coordinates in a right-handed coordinate system, $\hat{x} = (x_1, x_2)$, ∇^2 and $\hat{\nabla}^2$ are defined in §1, and satisfying the conditions

$$\epsilon^h(x,t)|_{t<0} = 0,$$

$$\epsilon^h(x,t) \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{for any fixed} \quad h \quad \text{and} \quad t.$$

Everywhere here we assume that $N^2(x_3)$ is smooth, bounded and positive.

When the buoyancy frequency is constant, $N(x_3) = N = \text{const}$, the fundamental solution has the form (see Dickinson 1969; Sekerzh-Zen'kovich 1979; Gabov & Sveshnikov 1986)

$$\epsilon^h(x,t) = E(|\hat{x}|, |x_3 - h|, t) \equiv -\frac{H(t)}{2\pi^2(|\hat{x}|^2 + (x_3-h)^2)^{\frac{1}{2}}}$$

$$\times \int_{d(\hat{x}, x_3-h)}^{N} \sin(\omega t)\,(N^2 - \omega^2)^{-\frac{1}{2}}(\omega^2 - d^2(\hat{x}, x_3 - h))^{-\frac{1}{2}}\,\mathrm{d}\omega \tag{2.2a}$$

or, equivalently,

$$E(|\hat{x}|, |x_3-h|, t) = -\frac{H(t)}{4\pi(|\hat{x}|^2+(x_3-h)^2)^{\frac{1}{2}}}\int_0^t J_0(N(t-\tau))\, J_0(d(\hat{x}, x_3-h)\,\tau)\,\mathrm{d}\tau \quad (2.2b)$$

where $H(t)$ is the Heaviside step function, $|\hat{x}|^2 = x_1^2+x_2^2$, $d(x_1, x_2, x_3) = d(x) = N|x_3|/|x|$ and J_0 is the Bessel function of order zero.

For arbitrary stratified fluids $\epsilon^h(x,t)$ has not been obtained in closed form.

Taking advantage of the Green's formulae technique, one can obtain a representation of the solution to the Cauchy problem, i.e. of the function $u(x,t)$ satisfying equation (1.1) in the whole space R^3 and the initial conditions $u|_{t=0} = u_0(x)$ and $u_{t|t=0} = u_1(x)$:

$$\begin{aligned} u(x,t) = &\int_{R^3}\int_0^t \mathrm{d}\tau\, \mathrm{d}^3\xi\, \epsilon^{x_3}(\hat{x}-\hat{\xi}, \xi_3, t-\tau) f(\xi,\tau) \\ &+\int_{R^3} \mathrm{d}^3\xi \{\epsilon^{x_3}(\hat{x}-\hat{\xi}, \xi_3, t)\, \nabla_\xi^2 u_1(\xi) \\ &+\epsilon_t^{x_3}(\hat{x}-\hat{\xi}, \xi_3, t)\, \nabla_\xi^2 u_0(\xi)\}. \end{aligned} \quad (2.3)$$

Here and below the subscript ξ indicates that the relevant operators are composed of derivatives with respect to ξ_i.

In the presence of boundaries one has to use Green's functions instead of the fundamental solution. We consider equation (1.1) outside a volume V with a piecewise smooth boundary S and impose the Dirichlet condition on the boundary: $u|_S = g(x,t)$, the condition at infinity: $|u| \to 0$ as $|x| \to \infty$, and initial conditions: $u|_{t=0} = u_0(x)$, $u_t|_{t=0} = u_1(x)$.

Let $G(\xi, t\,|\,x)$ be the solution of the problem:

$$\frac{\partial^2}{\partial t^2}\nabla_\xi^2 G(\xi,t\,|\,x) + N^2(\xi_3)\,\hat{\nabla}_\xi^2 G(\xi,t\,|\,x) = \delta(\xi-x,t), \quad (2.4a)$$

$$\left.\begin{aligned} &x, \xi \in R^3\backslash V, \\ &G(\xi,t\,|\,x)|_{\xi\in S} = 0, \quad G(\xi,t\,|\,x)|_{t<0} = 0, \\ &G(\xi,t\,|\,x) \to 0 \quad \text{as} \quad |\xi| \to \infty \quad \text{under any fixed } x \text{ and } t. \end{aligned}\right\} \quad (2.4b)$$

Using Green's theorems as above, one can derive the formula

$$\begin{aligned} u(x,t) = &\int_{R^3\backslash V}\int_0^t \mathrm{d}\tau\, \mathrm{d}^3\xi\, G(\xi, t-\tau\,|\,x) f(\xi,\tau) \\ &+\int_{R^3\backslash V} \mathrm{d}^3\xi \{G(\xi. t\,|\,x)\, \nabla_\xi^2 u_1(\xi) + G_t(\xi,t\,|\,x)\, \nabla_\xi^2 u_0(\xi)\} \\ &-\int_S \mathrm{d}^2 S_\xi \Big\{ g(\xi,t)\, \boldsymbol{n}_\xi \cdot \boldsymbol{\nabla}_\xi G_t(\xi, 0\,|\,x) \\ &+\int_0^t \mathrm{d}\tau\, g(\xi,\tau)\left(\boldsymbol{n}_\xi \frac{\partial^2}{\partial t^2} + N^2(\xi_3)\,\hat{\boldsymbol{n}}_\xi\right)\cdot \boldsymbol{\nabla}_\xi G(\xi, t-\tau\,|\,x)\Big\}. \end{aligned} \quad (2.5)$$

where $\boldsymbol{n}_\xi$ denotes the outward normal from V at ξ and $\hat{\boldsymbol{n}}_\xi = \boldsymbol{n}_\xi - \boldsymbol{e}_3(\boldsymbol{e}_3 \cdot \boldsymbol{n})$.

We remark that even in the case of constant buoyancy frequency the Green's functions cannot be found explicitly for arbitrary domains. A procedure for constructing the Green's function in the horizontal layer $H_0 < x_3 < H_1$ can be found in Anyutin & Borovikov (1986). This procedure leads to a series in the eigenfunctions of the following Sturm–Liouville problem:

$$\phi_n''(x_3)+\frac{\kappa^2}{\omega_n^2}(N^2(x_3)-\omega_n^2)\,\phi_n(x_3)=0,$$

$$\phi_n(\kappa,H_0)=\phi_n(\kappa,H_1)=0,$$

which, however, cannot be solved explicitly for arbitrary $N^2(x_3)$. The Green's function for the half-space $x_3>0$ in the case of a linear buoyancy frequency has been constructed by Borovikov (1988).

For domains having a certain shape and position it is possible to express $G(\xi,t\,|\,x)$ through the fundamental solution of (1.1). Let us present some domains and the corresponding Green's functions, using cylindrical coordinates (r,ϕ,x_3), where (r,ϕ) are polar coordinates on the plane $0x_1x_2$ with the pole at the origin and ϕ counted anticlockwise from $0x_1$.

(*a*) V is a spatial angle of the form:

$$V_a^n=\{x\,|\,\phi\in(0,\pi/n),\quad r\in(0,\infty),\quad x_3\in(-\infty\,;+\infty)\},\quad n\in\mathbb{N}.$$

Let $(\hat{x},x_3)\in V_a^n$. Reflecting $(\hat{x},x_3)$ symmetrically with respect to the half-plane $\phi=\pi/n$ we obtain a point $(\hat{x}_1,x_3)$. Then we transform $(\hat{x}_1,x_3)$ symmetrically with respect to the half-plane $\phi=2\pi/n$. We continue this procedure until we obtain the point $(\hat{x}_{2n-1},x_3)$, which is symmetrical to the point $(\hat{x}_{2n-2},x_3)$ with respect to the half-plane $\phi=(2n-1)\pi/n$, and to the point $(\hat{x},x_3)$ with respect to the half-plane $\phi=0$. It is clear that $\hat{x}_\iota$ are functions of $\hat{x}$, but for brevity we omit the demonstration of this dependence. The function

$$G_a^n(\xi,t\,|\,x)=\epsilon^{x_3}(\hat{x}-\hat{\xi},\xi_3,t)+\sum_{\kappa=1}^{2n-1}(-1)^\kappa\epsilon^{x_3}(\hat{x}_\kappa-\hat{\xi},\xi_3,t)\qquad(2.6a)$$

is easily seen to be the Green's function for equation (1.1) in V_a^n subject to the Dirichlet condition on the boundary of V_a^n.

(*b*) V is the half-space:

$$V_b=\{x\,|\,x_3>0\}.$$

The buoyancy frequency in this case is not defined for $x_3<0$. In order to avoid dealing with non-smooth $N^2(x_3)$ we assume in this subsection and in the next one, that its even extension (i.e. that which gives $N^2(x_3)=N^2(-x_3)$ for negative x_3) is smooth. Inserting this extended $N^2(x_3)$ into (2.1) we obtain the problem for the corresponding fundamental solution. The Green's functions is expressed through this fundamental solution as follows:

$$G_b(\xi,t\,|\,x)=\epsilon^{x_3}(\hat{x}-\hat{\xi},\xi_3,t)-\epsilon^{-x_3}(\hat{x}-\hat{\xi},\xi_3,t).\qquad(2.6b)$$

(*c*) V is a fluid layer, and the buoyancy frequency is constant:

$$V_c=\{x\,|\,0<x_3<H\},\quad N^2(x_3)=N^2=\text{const.}>0.$$

Consider the sequence $\{h_n(x_3)\}$ defined as follows:

$$\begin{aligned}h_{4n-3}&=-(x_3+2(n-1)H),\\ h_{4n-2}&=-(2nH-x_3),\\ h_{4n-1}&=(-x_3+2nH),\\ h_{4n}&=(2nH+x_3),\quad n\in\mathbb{N}\end{aligned}$$

The sequence $h_n(x_3)$ is easily seen to be none other than that of the vertical coordinates of points obtained by successively reflecting a point, whose vertical coordinate is $x_3 \in (0, H)$, with respect to the planes $x_3 = 0$ and $x_3 = H$.

Let us introduce the function

$$G_c(\xi, t \mid x) = E(|\hat{x}-\hat{\xi}|, |x_3-\xi_3|, t) + \sum_{n=1}^{\infty} (-1)^n E(|\hat{x}-\hat{\xi}|, |h_n(x_3)-\xi_3|, t). \tag{2.6c}$$

It can be shown, with the help of the Dirichlet test for convergence of a series, that the series on the right-hand side of (2.6*c*) converges uniformly with respect to $(\xi, t) \in Q \times [0, T]$ for any compactum $Q \subset V_c$ and any $T > 0$. The function $G_c(\xi, t \mid x)$ is easily seen to satisfy (2.4). Therefore $G_c(\xi, t \mid x)$ is the Green's function for equation (1.1) in V_c subject to the Dirichlet condition on the planes $x_3 = 0$ and $x_3 = H$.

In exactly the same way we can construct the appropriate Green's functions for the domain between two vertical parallel planes, it being possible in this case to do the derivation for a buoyancy frequency which varies with altitude. We do not dwell on this subject, but turn now to considering problems with monochromatic forcing.

3. Harmonic excitation

In the present section we investigate the behaviour of solutions as $t \to \infty$ under monochromatic excitation. In the main we restrict ourselves to considering monochromatic point sources, i.e. $f(x,t) = \delta(x-x_0)\,e^{i\sigma t}$, $\sigma > 0$, where the location x_0 is determined by the geometry of the relevant problem under consideration. Under zero initial and boundary conditions the solution assumes the form

$$u(x,t) = e^{i\sigma t}\int_0^t e^{-i\sigma\tau}\, G(x, \tau \mid x_0)\, d\tau, \tag{3.1}$$

where $G(x, \tau \mid x_0)$ is the corresponding Green's function (see (2.4)). In the case of the Cauchy problem with zero initial conditions the solution has the form:

$$u(x,t) = e^{i\sigma t}\int_0^t e^{-i\sigma\tau}\, \epsilon^h(\hat{x}-\hat{x}_0, x_3, \tau)\, d\tau, \tag{3.2}$$

where $(\hat{x}_0, h) = x_0$. Formulae (3.1) and (3.2) can be easily derived from the results of the previous section by taking advantage of the reciprocity property of the Green's functions and the fundamental solution:

$$G(x, t \mid x_0) = G(x_0, t \mid x), \quad \epsilon^h(\hat{x}, x_3, t) = \epsilon^{x_3}(\hat{x}, h, t).$$

We remark that the zero initial and boundary conditions mean that we need not investigate the behaviour of the last two integrals in (2.5) at large times. However, it is clear that for $u_0(x)$, $u_1(x)$ tending to zero sufficiently rapidly as $|x| \to \infty$, and $g(\xi, t)$ 'acting' during only a finite time interval (and vanishing fast enough as $|\xi| \to \infty$ if the domain V is unbounded), these terms tend to zero as $t \to \infty$ at the same rate as the corresponding Green's function (fundamental solution).

Now we proceed to investigate the behaviour of (3.2) at large times. Below we take the frequency of oscillations to satisfy the inequality $0 < \sigma < N(h)$, i.e. the corresponding steady oscillation equation to be hyperbolic. We shall investigate the behaviour of (3.2) as $t \to \infty$ at points of this equation's characteristic surface emanating from the point x_0 and begin with the case of the constant buoyancy frequency. Note that for this case the $t \to \infty$ behaviour of (3.2) has already been thoroughly investigated at points not belonging to the characteristic surface (see Gabov & Sveshnikov 1986, p. 103, and Voisin 1991).

(*a*) $N(x_3) = N = \text{const}$. The characteristic surface is the cone:

$$K^\sigma(x_0) = \left\{ x \mid \sigma = d(x-x_0) \equiv N\frac{|x_3-h|}{|x-x_0|} \right\}.$$

The Laplace transform of (2.2) has the form

$$\hat{E}(x-x_0, p) = -[4\pi|x-x_0|((p^2+N^2)(p^2+d^2(x-x_0))^{\frac{1}{2}}]^{-1}, \tag{3.3}$$

where the branch of the square root is chosen so that it assumes positive values on the real axis of p and has the following cuts:

$$(-\infty \pm \mathrm{i}N; \pm \mathrm{i}N], \quad (-\infty \pm \mathrm{i}d(x-x_0); \pm \mathrm{i}d(x-x_0)]$$

Let x be such that $d(x-x_0) = \sigma$, i.e. $x \in K^\sigma(x_0)$. Taking advantage of the formula for the inverse Laplace transform and the convolution property of the Laplace transform we find that (3.2) takes the form:

$$u(x,t) = -(8\pi^2\mathrm{i}\,|x-x_0|)^{-1}\int_{b-\mathrm{i}\infty}^{b+\mathrm{i}\infty} \mathrm{e}^{pt}((p^2+N^2)(p^2+\sigma^2))^{-\frac{1}{2}}(p-\mathrm{i}\sigma)^{-1}\,\mathrm{d}p,$$
$$b > 0, \quad x \in K^\sigma(x_0).$$

Consider now the integral

$$Q^\sigma(t) = -\int_{b-\mathrm{i}\infty}^{b+\mathrm{i}\infty} \mathrm{e}^{pt}[((p^2+N^2)(p^2+\sigma^2))^{-\frac{1}{2}}(p-\mathrm{i}\sigma)^{-1}$$
$$-(N^2-\sigma^2)^{-\frac{1}{2}}(2\sigma)^{-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\pi/4}(p-\mathrm{i}\sigma)^{-\frac{3}{2}}]\,\mathrm{d}p, \quad b > 0,$$

where the branch $(p-\mathrm{i}\sigma)^{-\frac{3}{2}}$ assumes positive values on the ray $(\mathrm{i}\sigma, \mathrm{i}\sigma+\infty)$ and has the cut $(\mathrm{i}\sigma-\infty, \mathrm{i}\sigma)$. Using Cauchy's theorem we can represent $Q^\sigma(t)$ as the integral over the cuts of the integrand and obtain the estimate: $|Q^\sigma(t)| < C(\sigma)\,t^{-\frac{1}{2}}$. Taking advantage of the fact that the Laplace transform of $t^{\frac{1}{2}}$ is $\pi^{\frac{1}{2}}/2p^{\frac{3}{2}}$ we get the formula

$$u(x,t) = \frac{1}{4\pi|x-x_0|}\left(\frac{\sqrt{2}\mathrm{e}^{\mathrm{i}(\sigma t+3\pi/4)}}{(\sigma\pi)^{\frac{1}{2}}(N^2-\sigma^2)^{\frac{1}{2}}}t^{\frac{1}{2}} + Q^\sigma(t)\right), \tag{3.4}$$

where $|Q^\sigma(t)| < C(\sigma)\,t^{-\frac{1}{2}}$ and $x \in K^\sigma(x_0)$. We shall comment on this result after case (*b*) of the present section, where the case of a buoyancy frequency varying with altitude is considered.

(*b*) $N(x_3) \neq \text{const}$, $0 < \sigma < \inf N(x_3)$.

Let $\tilde{\epsilon}^h(\hat{x}-\hat{x}_0, x_3, \omega)$ denote the Fourier transform of $\epsilon^h(\hat{x}-\hat{x}_0, x_3, \tau)$ with respect to τ. We can rewrite (3.2) as follows:

$$u(x,t) = \frac{\mathrm{e}^{\mathrm{i}\sigma t}}{2\pi}\int_0^t \mathrm{e}^{-\mathrm{i}\sigma\tau}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\omega\tau}\,\tilde{\epsilon}^h(\hat{x}-\hat{x}_0, x_3, \omega)\,\mathrm{d}\omega\,\mathrm{d}\tau. \tag{3.5}$$

It is clear that, as $t \to \infty$, the integral on the right tends to $\tilde{\epsilon}^h(\hat{x}-\hat{x}_0, x_3, \sigma)$, which is a fundamental solution of the steady oscillation equation, i.e.

$$(-\sigma^2\nabla^2 + N^2(x_3)\hat{\nabla}^2)\,\tilde{\epsilon}^h(\hat{x}-\hat{x}_0, x_3, \sigma) = \delta(x-x_0). \tag{3.6}$$

Using work by Hadamard (1932), we extract from $\tilde{\epsilon}^h(\hat{x}-\hat{x}_0, x_3, \sigma)$ a function containing its all main singularities and having the form

$$g^h(\hat{x}-\hat{x}_0, x_3, \sigma) = -(4\pi)^{-1}(N^2(x_3)-\sigma^2)^{-\frac{1}{4}}(N^2(h)-\sigma^2)^{-\frac{1}{4}}\Gamma^{-\frac{1}{2}}(\sigma, x, x_0), \tag{3.7}$$

where Γ is the square of the geodesic distance between the points x and x_0:

$$\Gamma(\sigma, x, x_0) = \phi^2(\sigma, x_3, h) - \sigma^2 |\hat{x} - \hat{x}_0|^2,$$

$$\phi(\sigma, x_3, h) = \int_h^{x_3} (N^2(z) - \sigma^2)^{\frac{1}{2}} \, dz,$$

and the argument of the complex number $\Gamma^{\frac{1}{2}}$ when $\Gamma < 0$ is equal to $\frac{1}{2}\pi$. Function (3.7) is the first term of the Hadamard expansion of $\tilde{e}^h(\hat{x} - \hat{x}_0, x_3, \sigma)$ in powers of Γ, the next term being proportional to $\Gamma^{\frac{1}{2}}$. It can be shown that $g^h(\hat{x} - \hat{x}_0, x_3, \sigma)$ coincides with $\tilde{e}^h(\hat{x} - \hat{x}_0, x_3, \sigma)$ when $N(x_3)$ is constant.

Let

$$x \in K^\sigma(x_0) \overset{\text{def}}{=} \{x \mid \Gamma(\sigma, x, x_0) = 0, \quad x \neq x_0\}$$

(this definition of $K^\sigma(x_0)$ is easily seen to include the one given for case (*a*) above. Changing the order of integration in (6.5) and integrating over τ gives

$$u(x, t) = \frac{e^{i\sigma t}}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i(\omega - \sigma)t} - 1}{\omega - \sigma} \tilde{e}^h(\hat{x} - \hat{x}_0, x_3, \omega) \, d\omega, \quad x \in K^\sigma(x_0).$$

Using (3.6) one can show that $\tilde{e}^h(\hat{x} - \hat{x}_0, x_3, \omega)$ vanishes as ω^{-2} as $|\omega| \to \infty$ for any fixed x, x_0. Taking advantage of this fact and the properties of $g^h(\hat{x} - \hat{x}_0, x_3, \sigma)$ mentioned above we can write

$$u(x, t) = \frac{e^{i\sigma t}}{2\pi i} \int_{\sigma - \epsilon}^{\sigma + \epsilon} \frac{e^{i(\omega - \sigma)t} - 1}{\omega - \sigma} g^h(\hat{x} - \hat{x}_0, x_3, \omega) \, d\omega + R(x, x_0, \sigma, t), \quad x \in K^\sigma(x_0), \tag{3.8}$$

where $|R(x, x_0, \sigma, t)| < C(x, x_0, \sigma)$ and ϵ is sufficiently small. From (3.7), (3.8) one can derive the formula

$$u(x, t) = \frac{\sqrt{2}\, e^{i(\sigma t + 3\pi/4)} t^{\frac{1}{2}}}{4\pi(\sigma\pi)^{\frac{1}{2}} \chi(x, x_0, \sigma)} + R_1(x, x_0, \sigma, t), \tag{3.9}$$

where $x \in K^\sigma(x_0)$,

$$\chi(x, x_0, \sigma) = (N^2(x_3) - \sigma^2)^{\frac{1}{4}} (N^2(h) - \sigma^2)^{\frac{1}{4}} \times \left\{ \int_h^{x_3} (N^2(z) - \sigma^2)^{-\frac{1}{2}} dz \int_h^{x_3} (N^2(\eta) - \sigma^2)^{\frac{1}{2}} d\eta + |\hat{x} - \hat{x}_0|^2 \right\}^{\frac{1}{2}}$$

and

$$|R_1(x, x_0, \sigma, t)| < C(x, x_0, \sigma).$$

The principal term of (3.9) looks like the principal term of (3.4) and it is easily checked that they become the same when the buoyancy frequency is constant.

Formulae (3.4) and (3.9) show that the solution of the Cauchy problem for a monochromatic point source, whose frequency is less than the buoyancy frequency, diverges as $t^{\frac{1}{2}} e^{i\sigma t}$ when $t \to \infty$ on the cone $K^\sigma(x_0)$, which is in fact the steady oscillation equation's characteristic surface emanating from the point source. The oscillations with growing amplitude on the cone $K^\sigma(x_0)$ have a phase shift of $\frac{3}{4}\pi$ with respect to the phase of the point source, while the corresponding phase shifts of oscillations at points inside and outside the cone can be shown to equal π and $\frac{1}{2}\pi$ respectively. The presence of boundaries complicates the manifold on which the growth of the amplitude takes place. In the examples (*a*)–(*c*) of §2 each of these manifolds is composed as is easily seen from the corresponding Green's function $G(\xi, t \mid x)$, of the intersection of the relevant V with the union of $K^\sigma(x_0)$, $K^\sigma(x_{0i})$ where x_{0i} denote the relevant reflections of x_0 described in (a)–(c) of §2. The oscillations' phase pattern is also complicated by the presence of boundaries, but the phase shift of the oscillations with growing amplitude on the manifolds equals either $(-\frac{1}{4}\pi)$ or $\frac{3}{4}\pi$.

In the work by Borovikov (1990) a hypothesis about the asymptotic behaviour of $\epsilon^h(\hat{x}-\hat{x}_0, x_3, t)$ for $t \to \infty$ is proposed. An application of this hypothesis to our subject gives a result analogous to that obtained above. Indeed, according to Borovikov's paper the principal term $p^h(\hat{x}-\hat{x}_0, x_3, t)$ of the asymptotic expansion at a point x is formed from contributions from the rays passing through x. Let x be a point on a ray with $\omega = \sigma$ (i.e. $x \in K^\sigma(x_0)$). The contribution of the ray to $p^h(\hat{x}-\hat{x}_0, x_3, t)$ has the form $-A(x, x_0)\, t^{-\frac{1}{2}} \sin(\sigma t + \frac{1}{4}\pi)$. If x does not lie on a line at which rays turn, then

$$p^h(\hat{x}-\hat{x}_0, x_3, t) = -A(x, x_0)\, t^{-\frac{1}{2}} \sin(\sigma t + \tfrac{1}{4}\pi) + \sum_\kappa A_\kappa(x, x_0)\, t^{-\frac{1}{2}} \sin(\omega_\kappa t + \psi_\kappa),$$

where the sum is taken over other rays passing through x and having $\omega = \omega_\kappa \mp \sigma$, the residual

$$B(x, x_0, t) \equiv \epsilon^h(\hat{x}-\hat{x}_0, x_3, t) - p^h(\hat{x}-\hat{x}_0, x_3, t)$$

satisfying the estimate

$$|B(x, x_0, t)| < C(x, x_0)\, t^{-\frac{3}{2}}.$$

Simple calculations and estimates show that (3.2) can be represented as follows:

$$u(x,t) = A(x, x_0)\, e^{i(\sigma t + 3\pi/4)}\, t^{\frac{1}{2}} + q(x, x_0, t), \tag{3.10}$$

where $|q(x, x_0, t)| \leqslant C_1(x, x_0)$.

Formula (3.10) is easily seen to be in agreement with (3.9).

Borovikov's method also enables one to investigate the phenomenon under consideration in the presence of a critical level, at which $N(x_3) = \sigma$. However, it is not applicable, or needs more accurate treatment, when an infinite number of rays passes through x. (The author thanks a referee, who drew the author's attention to this fact.)

One can also consider the forcing

$$f(x,t) = \delta(x - x_0)\, \phi(t)\, e^{i\sigma t},$$

where $\phi(t)$ is continuous and $\phi(t) = t^{-\alpha}$ starting from the moment $T > 0$, for some $\alpha \in (0,1)$. Suppose that $N = \text{const}$ and $\sigma \in (0, N)$. Representing the solution corresponding to this forcing through the inverse Laplace transform of the convolution in time of $E(|\hat{x}-\hat{x}_0|, |x_3 - h|, t)$ and $\phi(t)\, e^{i\sigma t}$ and using the inverse Laplace transform of $(p - i\sigma)^{\alpha - \frac{3}{2}}$, one can find that the principal term of the asymptotic expansion of $u(x,t)$ on the cone $K^\sigma(x_0)$ is of the form

$$u(x,t) \sim \frac{e^{i(\sigma t + 3\pi/4)}}{4\pi |x - x_0|}\, t^{\frac{1}{2}-\alpha} R(\sigma, \alpha), \quad x \in K^\sigma(x_0),$$

where

$$R(\sigma, \alpha) = \frac{\Gamma(1-\alpha)}{\Gamma(\frac{3}{2}-\alpha)} (N^2 - \sigma^2)^{-\frac{1}{4}} (2\sigma)^{-\frac{1}{2}},$$

$\alpha \in (0,1)$, and $\Gamma(z)$ is Euler's gamma function.

We remark that the next order terms are $O(t^{-\frac{1}{2}})$ and they must be taken into account when $\alpha \geqslant 1$.

Finally, let us consider a model of an extended monochromatic source, which illustrates a phenomenon of the focusing singularities kind. This phenomenon consists in the fact that at the apexes of cones of the limiting amplitude's singularities the divergence in time occurs at a 'higher' rate than at other points of the cones. As a result the character of the singularity of the limiting amplitude at the apexes of the cones differs from that at other points of the cones (see Appleby & Crighton 1987). We consider the Cauchy problem for equation (1.1) with zero initial conditions and the following forcing:

$$f(x,t) = \frac{1}{2\pi R}\int_{C_R} \delta(x-\xi)\,\mathrm{d}l_\xi\,\mathrm{e}^{\mathrm{i}\sigma t}, \tag{3.11}$$

where

$C_R = \{\xi \equiv (\hat{\xi}, \xi_3) \,|\, \xi_3 0,\ |\hat{\xi}| = R\}$ and $\mathrm{d}l_\xi$ is the element of length of C_R at $\xi \in C_R$.

Function (3.11) *simulates a ring-shaped monochromatic source.*

Let $N(x_3) = N = \text{const}$ and $\sigma \in (0, N)$. The solution of the problem has the form

$$u(x,t) = \frac{\mathrm{e}^{\mathrm{i}\sigma t}}{2\pi R}\int_0^t \mathrm{e}^{-\mathrm{i}\sigma\tau}\int_{C_R} E(|\hat{x}-\hat{\xi}|, |x_3-\xi_3|, \tau)\,\mathrm{d}l_\xi\,\mathrm{d}\tau, \tag{3.12}$$

where E is given by (2.2).

Let us now observe the behaviour of the solution as $t \rightarrow \infty$at points $0^\pm = (0, 0, \pm R\sigma(N^2-\sigma^2)^{-\frac{1}{2}})$, which are the apexes of characteristic cones containing C_R:

$$u(0^\pm, t) = \mathrm{e}^{\mathrm{i}\sigma t}\int_0^t \mathrm{e}^{-\mathrm{i}\sigma\tau} E(R, R\sigma(N^2-\sigma^2)^{-\frac{1}{2}}, \tau)\,\mathrm{d}\tau.$$

By virtue of (3.4) we have

$$u(0^\pm, t) = \frac{1}{4\pi RN}\left(\frac{\surd 2\,\mathrm{e}^{\mathrm{i}(\sigma t+3\pi/4)}}{(\sigma\pi)^{\frac{1}{2}}} t^{\frac{1}{2}} + Q^\sigma(t)\right),$$

where $|Q^\sigma(t)| < C(\sigma)\,t^{-\frac{1}{2}}$.

Therefore the ring-shaped source creates a velocity field, whose vertical component at points $0^\pm$ diverges as $t^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\sigma t}$ when $t \rightarrow \infty$. On the other hand, slightly more cumbersome calculations (contained in an Appendix†) show that at other points of the cones $K^\sigma(0^\pm)$, $u(x,t)$ diverges as $\mathrm{e}^{\mathrm{i}\sigma t}\ln t$. Thus, at points $O^\pm$ the focusing of singularities takes place.

For the variable buoyancy frequency $N(x_3)$ of case (*b*) of §2, the focusing of singularities still takes place, but at

$$\tilde{0}^\pm = (0, 0, a^\pm(R)),$$

where $a^\pm(R)$ are solutions of the equations

$$R = \int_0^{a^+} (N^2(z)-\sigma^2)^{\frac{1}{2}}\sigma^{-1}\,\mathrm{d}z,$$

$$R = \int_{a^-}^{0} (N^2(z)-\sigma^2)^{\frac{1}{2}}\sigma^{-1}\,\mathrm{d}z.$$

The principal term of the asymptotic expansion of $u(x,t)$ at $\tilde{0}^\pm$ is given by (3.9) with $x_3 = a^\pm$, $h = 0$ and $|\hat{x}-\hat{x}_0| = R$.

† Available on request from the author or the Editorial Office.

4. Conclusions

The problem of the appearance of singularities in the amplitude of oscillations in stratified fluids can be investigated in the time-dependent formulation proceeding from equation (1.1). The structure of the equation enables us to express its Green's functions for certain domains through its fundamental solution. This reduces the problem in the case of a monochromatic point source to investigating integrals of the form (3.2). In the absence of critical levels, the $t \to \infty$ behaviour of the vertical component of the velocity field for an impulsively started monochromatic point source on $K^{\sigma}(x_0)$ is given by (3.9). The approach used also enables us to observe phenomena like focusing singularities by considering a monochromatic ring-shaped source (3.10) or any of its parts of finite length.

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